Problem 13) a) The derivative of $\sinh x$ is $\cosh x$, while the derivative of $\cosh x$ is $\sinh x$. At $x=0$, we have $\sinh(0)=0$ and $\cosh(0)=1$. Therefore,

Taylor series: $f(x)=f(0)+f'(0)x+\frac{1}{2!}f''(0)x^2+\frac{1}{3!}f'''(0)x^3+\cdots$

$$\sinh x = x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$$\cosh x = 1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

b) The defining property of an even function is $f(x)=f(-x)$; consequently, $f'(x)=-f'(-x)$. The only way to satisfy this equation at $x=0$ is to have $f'(0)=0$. Taking the next derivative, we find $f''(x)=f''(-x)$, which does not impose any constraints on the value of $f''(0)$. However, the next derivative gives $f'''(x)=-f'''(-x)$, which requires that $f'''(0)=0$. It is thus seen that the 1st, 3rd, 5th, 7th, $\cdots$ derivatives of $f(x)$ at $x=0$ are all equal to zero. Consequently, the Taylor series expansion of $f(x)$ is comprised only of even powers of x.

The defining property of an odd function is $g(x)=-g(-x)$; consequently, $g'(x)=g'(-x)$, and $g''(x)=-g''(-x)$. The only way to satisfy the last equation at $x=0$ is to have $g''(0)=0$. The next derivative gives $g'''(x)=g'''(-x)$, which does not impose any constraints on the value of $g'''(0)$. Taking the next derivative, however, gives $g''''(x)=-g''''(-x)$, which requires $g''''(0)=0$. It is thus seen that the 2nd, 4th, 6th, 8th, $\cdots$ derivatives of $g(x)$ at $x=0$ are all equal to zero. Consequently, the Taylor series expansion of $g(x)$ is comprised only of odd powers of x.

c) The function $\tanh(x)$ is an odd function of x, because $\sinh(x)$ is odd while $\cosh(x)$ is even. The Taylor series expansion of $\tanh(x)$ may thus be written as $\tanh(x)=\sum_{n=0}^{\infty}a_n x^{2n+1}$.

d)
$$\tanh(x)=\sum_{n=0}^{\infty}a_n x^{2n+1}=\left[\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\right]\Big/\left[\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\right]$$

$$\rightarrow\quad \left(\sum_{n=0}^{\infty}a_n x^{2n+1}\right)\left[\sum_{m=0}^{\infty}\frac{x^{2m}}{(2m)!}\right]=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$ ← Changing a dummy index from n to m.

$$\rightarrow\quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{a_n}{(2m)!}x^{2(n+m)+1}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

$$\rightarrow\quad \sum_{k=0}^{\infty}\left[\sum_{m=0}^{k}\frac{a_{k-m}}{(2m)!}\right]x^{2k+1}=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$ ← Summing along diagonals in the mn-plane; setting $k=m+n$.

$$\rightarrow\quad \sum_{m=0}^{n}\frac{a_{n-m}}{(2m)!}=\frac{1}{(2n+1)!}\qquad (n=0,1,2,3,\ldots).$$ ← Changing dummy index k back to n.

The first few coefficients of the Taylor series expansion of $\tanh(x)$ around $x=0$ are found to be

$$n=0\quad\rightarrow\qquad a_0=1;$$

$$n=1\quad\rightarrow\quad \frac{a_1}{0!}+\frac{a_0}{2!}=\frac{1}{3!}\quad\rightarrow\quad a_1=-\frac{1}{3};$$

$$n=2\quad\rightarrow\quad \frac{a_2}{0!}+\frac{a_1}{2!}+\frac{a_0}{4!}=\frac{1}{5!}\quad\rightarrow\quad a_2=\frac{2}{15};$$

$$n=3\quad\rightarrow\quad \frac{a_3}{0!}+\frac{a_2}{2!}+\frac{a_1}{4!}+\frac{a_0}{6!}=\frac{1}{7!}\quad\rightarrow\quad a_3=-\frac{17}{315}.$$